Note that the Coriolis coupling results in a 90° phase shift between the w and v components of displacement. The separated equations for free vibration then become

$$\left[\frac{(EI)_w}{\Omega^2R^4}w''\right]'' - \left[\left(\int_x^1 m\eta d\eta\right)w'\right]' - m(\nu^2 w + 2\nu\beta v) = 0 \quad (10)$$

$$\left[\frac{(EI)_v}{\Omega^2R^4}v''\right]'' - \left[\left(\int_x^1 m\eta d\eta\right)v'\right]' - m[(\nu^2+1)v + 2\nu\beta w] = 0 \quad (11)$$

The appropriate boundary conditions are the same as for the uncoupled case of a cantilever blade

$$v(0) = w(0) = v'(0) = w'(0) = v''(1) = w''(1) = v'''(1) = w'''(1) = 0 \quad (12)$$

Equations (10) and (11) are now solved by Galerkin's method,[4] utilizing the known, uncoupled modes of vibration which are also the solutions for no preconing or vanishing β. This permits a direct comparison and evaluation of the significance of the Coriolis coupling. Let

$$w(x) \cong \sum_{i=1}^{n} a_i w_i(x) \quad (13)$$

$$v(x) \cong \sum_{i=1}^{n} b_i v_i(x) \quad (14)$$

Associated with w_i and v_i, the ith modes of uncoupled out-of-plane and in-plane vibrations are the uncoupled natural frequency ratios ν_{wi} and ν_{vi}, respectively. Substituting Eqs. (13) and (14) into Eqs. (10) and (11), multiplying through by w_j and v_j, respectively, and integrating over the dimensionless span there results the $2n$ homogeneous algebraic equations

$$\int_0^1 mw_j \sum_{i=1}^{n} [a_i(\nu^2 - \nu_{wi}^2)w_i + 2\nu\beta b_i v_i]dx = 0, \quad j = 1,2,\ldots,n \quad (15)$$

$$\int_0^1 mv_j \sum_{i=1}^{n} [b_i(\nu^2 - \nu_{vi}^2)v_i + 2\nu\beta a_i w_i]dx = 0, \quad j = 1,2,\ldots,n \quad (16)$$

Limiting our attention in this analysis to the fundamental modes of coupled vibration, the series expansions in the Galerkin solution are truncated to

$$w \cong a_1 w_1 \text{ and } v \cong b_1 v_1 \quad (17)$$

This results in the biquadratic characteristic equation

$$\left(\frac{\nu}{\nu_{w1}}\right)^4 - \left[1 + \left(\frac{\nu_{v1}}{\nu_{w1}}\right)^2 + \left(\frac{2\beta\gamma}{\nu_{w1}}\right)^2\right]\left(\frac{\nu}{\nu_{w1}}\right)^2 + \left(\frac{\nu_{v1}}{\nu_{w1}}\right)^2 = 0 \quad (18)$$

$$\gamma^2 = \left(\int_0' mw_1v_1dx\right)^2 \Big/ \left(\int_0' mw_1^2dx\right)\left(\int_0' mv_1^2dx\right) \quad (19)$$

where the coupled frequency ratio ν is normalized by the uncoupled out-of-plane frequency ratio ν_{w1}, and is seen to depend on two parameters: first, the ratio of the in-plane to out-of-plane fundamental uncoupled bending frequencies and a modal coupling factor proportional to the preconing angle.

Sample Calculation

To illustrate the potential importance of the Coriolis coupling, a sample calculation has been carried out for a typical fundamental uncoupled out-of-plane frequency ratio[5] $\nu_{w1} = 1.11$ and an uncoupled in-plane frequency ratio of $\nu_{v1} = 0.70$ which is typical of a helicopter designed with in-plane flexural rigidity close to that for out-of-plane bending.[6] The precone angle β is taken as 6°, corresponding to a heavy aerodynamic loading from Eq. (6). The modal coupling factor γ is approximated as unity. This results in the coupled frequency ratios $\nu = 1.14$ and $\nu = 0.68$.

Concluding Remarks

The relatively small numerical differences between the coupled and uncoupled frequencies in the sample calculation can be significant in the flying qualities and ground resonance stability of the helicopter. In the case of the higher frequency (dominantly out-of-plane bending) the increase in frequency caused by Coriolis coupling will tend to increase rotor control power.[5] In the case of the lower frequency (dominantly inplane bending), the dynamic stability of the system on the ground ("ground resonance") is sensitive to small changes.[6] Perhaps of equal importance in this case is that the out-of-plane component caused by Coriolis coupling contributes some aerodynamic coupling which should be included in the ground resonance analysis of hingeless rotor helicopters.

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Kernel Function for Nonplanar Oscillating Surfaces in Supersonic Flow

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RODEMICH[1] and Landahl[2] have given the subsonic acceleration potential kernel for nonplanar configurations in a form that has been rewritten by Rodden, Giesing, and Kalman[3] as

$$K = \exp(-i\omega x_0/U)(K_1T_1/r^2 + K_2T_2^*/r^4) \quad (1)$$

where ω is the frequency, x_0 is the distance between the sending and receiving points parallel to the freestream, U is the velocity of the freestream,

$$T_1 = \cos(\gamma_r - \gamma_s) \quad (2)$$

$$T_2^* = (z_0 \cos\gamma_r - y_0 \sin\gamma_r)(z_0 \cos\gamma_s - y_0 \sin\gamma_s) \quad (3)$$

$$r = (y_0^2 + z_0^2)^{1/2} \quad (4)$$

y_0 and z_0 are the Cartesian distances perpendicular to the freestream between the sending and receiving points, and γ_r and

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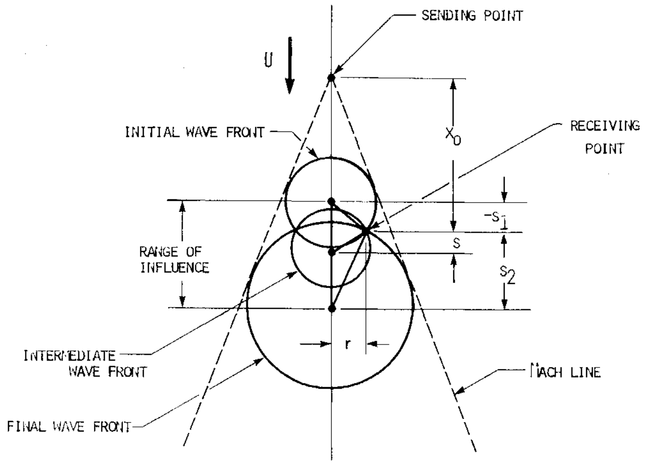

Fig. 1 Geometry for wave propagation to receiving point from wake of oscillating pressure disturbance at sending point in supersonic case.

γ_s are the dihedral angles at the receiving and sending points, respectively. The so-called planar and nonplanar parts of the kernel numerator have been given by[1]

$$K_1 = r(\partial I_0/\partial r) \tag{5}$$

and

$$K_2 = r^3(\partial/\partial r)[(1/r)\partial I_0/\partial r] \tag{6}$$

in which the integral I_0 is now rewritten to generalize its limits to read

$$I_0 = \int_{s_1}^{s_2} \frac{\exp(-i\omega s/U)}{(r^2+s^2)^{1/2}}\, ds \tag{7}$$

This integral may be interpreted as a potential at the receiving point from the waves emanating from the wake of the oscillating pressure disturbance at the sending point. Accordingly, the limits s_1 and s_2 are the distances along the wake and downstream from the receiving point from which the first and last waves, respectively, reach the receiving point as shown in Fig. 1. Equating the convection and propagation times, the limits s_1 and s_2 are then the extremes for which the inequality

$$(s + x_0)/U > (r^2 + s^2)^{1/2}/a \tag{8}$$

or

$$s + x_0 > M(r^2 + s^2)^{1/2} \tag{9}$$

is satisfied, where a and M are the speed of sound and Mach number, respectively.

Substituting Eq. (7) into Eqs. (5) and (6) yields

$$K_1 = \frac{r\exp(-i\omega s/U)}{(r^2+s^2)^{1/2}} \frac{\partial s}{\partial r}\Bigg|_{s=s_1}^{s=s_2} - \int_{s_1}^{s_2} \frac{r^2\exp(-i\omega s/U)}{(r^2+s^2)^{3/2}}\, ds \tag{10}$$

$$K_2 = \exp(-i\omega s/U)\left\{\frac{r}{(r^2+s^2)^{1/2}}\left[r\frac{\partial^2 s}{\partial r^2} - \frac{\partial s}{\partial r} - i\frac{\omega r}{U}\left(\frac{\partial s}{\partial r}\right)^2\right] - \frac{r^2}{(r^2+s^2)^{3/2}}\left[2r\frac{\partial s}{\partial r} + s\left(\frac{\partial s}{\partial r}\right)^2\right]\right\}\Bigg|_{s=s_1}^{s=s_2} + 3\int_{s_1}^{s_2}\frac{r^4\exp(-i\omega s/U)}{(r^2+s^2)^{5/2}}\, ds \tag{11}$$

The evaluation of K_1 and K_2 from Eqs. (10) and (11) depends on the Mach number regime. At subsonic speeds, the extremes of Eq. (9) are

$$s_1 = (MR - x_0)/\beta^2 \tag{12}$$

$$s_2 = \infty \tag{13}$$

where

$$R = (x_0^2 + \beta^2 r^2)^{1/2} \tag{14}$$

and

$$\beta^2 = 1 - M^2 \tag{15}$$

and Landahl's results[2] follow except for a reversal in sign. Rodden et al.[3] are seen to have been right in reversing the signs of Landahl's expressions, but for the wrong reason.

At supersonic speeds, the extremes of Eq. (9) are

$$s_1 = (x_0 - MR)/B^2 \tag{16a}$$

$$s_1 = (M^2r^2 - x_0^2)/(x_0 + MR) \tag{16b}$$

$$s_2 = (x_0 + MR)/B^2 \tag{17}$$

where

$$B^2 = -\beta^2 \tag{18a}$$

$$B^2 = M^2 - 1 \tag{18b}$$

for $x_0 > Br$, since disturbances are restricted to the region of their aft Mach cone. Equation (16b) provides a preferable computational form near sonic speed since, for $M = 1$,

$$s_1 = (r^2 - x_0^2)/2x_0 \tag{19}$$

$$s_2 = \infty \tag{20}$$

The supersonic kernel functions follow from Eqs. (10), (11), (16) and (17) and we obtain

$$K_1 = -u(x_0 - Br)\left\{\frac{Mr^2}{R}\left[\frac{\exp(-i\omega s_2/U)}{(r^2+s_2^2)^{1/2}} + \frac{\exp(-i\omega s_1/U)}{(r^2+s_1^2)^{1/2}}\right] + \int_{s_1}^{s_2}\frac{r^2\exp(-i\omega s/U)}{(r^2+s^2)^{3/2}}\, ds\right\} \tag{21}$$

and

$$K_2 = u(x_0 - Br)\left\{\frac{Mr^4\exp(-i\omega s_2/U)}{R(r^2+s_2^2)^{3/2}}\left[2 - \frac{B^2(r^2+s_2^2)}{R^2} - \frac{Ms_2}{R}\right] - i\frac{M^2r^3(\omega r/U)\exp(-i\omega s_2/U)}{R^2(r^2+s_2^2)^{1/2}} + \frac{Mr^4\exp(-i\omega s_1/U)}{R(r^2+s_1^2)^{3/2}}\left[2 - \frac{B^2(r^2+s_1^2)}{R^2} + \frac{Ms_1}{R}\right] + i\frac{M^2r^3(\omega r/U)\exp(-i\omega s_1/U)}{R^2(r^2+s_1^2)^{1/2}} + 3\int_{s_1}^{s_2}\frac{r^4\exp(-i\omega s/U)}{(r^2+s^2)^{5/2}}\, ds\right\} \tag{22}$$

in which we have introduced the unit step function $u(x_0 - Br)$ to show that the kernels vanish identically for $x_0 < Br$. The planar supersonic kernel function of Watkins and Berman[4] is obtained from Eq. (21) after integrating the integral by parts and setting $r = |y_0|$. The approximation of Laschka,[5] Eq. (53), for $u/(1 + u^2)^{1/2}$ may be used as in Ref. 3 for accurate evaluation of the integrals in Eqs. (21) and (22).

The successes of the Subsonic Doublet-Lattice Method[3,6,7] for oscillatory motion and the Constant Pressure Panel Method of Carmichael and Woodward[8,9] for steady subsonic and supersonic flows suggest the possibility of a supersonic finite-element method for interfering nonplanar configurations. The expressions for the kernel functions given here provide the basis, and the prospects for such a method are currently under investigation by the present authors.

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A Nonvarying-C^* Control Scheme for Aircraft

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Introduction

THE usual method for accommodating an automatic flight control system to the wide variations in dynamic characteristics of the airframe, with changes of airspeed and altitude over its flight envelope, is to change control-loop gains with measurements of air data. The dynamic pressure, or the Mach number, as estimated by an air data computer, is generally used as the parameter with which the gains are scheduled. Our system provides a response that is uniform for all flight conditions without using measurements of dynamic pressure, altitude or angle of attack. The scheme is very close to that used for SIDAC.[1,2] The modification is based on the observation that the equation for handling qualities C^* criterion is very similar to a basic short-period equation for the motion of the aircraft. A modest feedback and feedforward with variable gains holds the coefficients of this equation fixed. The C^* requirement may be met by choosing these coefficients to be the same as demanded by the Criterion or by adding a fixed outer loop. The mechanism for varying the gains is found by a gradient calculation similar to that used in the SIDAC analysis. We achieve the advantage of a low-gain, narrow-bandwidth system which is very insensitive to instrument nose and bending modes, accommodates the primary control system, and satisfies the requirements directly rather than using a model-following technique.

The SIDAC system identifies parameters and uses this in turn to adjust gains. Since the accuracy of identifying the several coefficients depends on the frequency content of the motion and the particular flight condition, it appears that a system which calls for the required response directly should have an advantage. This has also been argued by Hofmann and Best[3] in a fairly similar approach to control of the lateral-directional axes.

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C^*-Criterion

The most difficult problem in flight control design, besides considerations of making the system invulnerable to component failures, is in deciding on what flying characteristics will be acceptable to the pilot. The requirement for longitudinal response which best fits into an analytical formulation is that promulgated by Tobie, Elliot, and Malcom.[4] The criterion is that the time-response trace of a quantity called C^*, for an abrupt force applied to the stick, must fall within a certain envelope. C^* is the sum of the normal force applied to the pilot's seat plus a constant multiple of the angular velocity in pitch. Thus, it is a linear combination, with constant positive coefficients, of normal acceleration at the aircraft's center of gravity, the pitch velocity and the pitch acceleration, assuming the pilot's station is ahead of the center of gravity.

Development

We begin with the equations of perturbations from straight and level flight written as

$$\begin{aligned} \ddot{\theta} &= M_q\dot{\theta} + M_\alpha\alpha + M_{\dot{\alpha}}\dot{\alpha} + M_\delta\delta \\ n &= U_0(\dot{\theta} - \dot{\alpha}) = -Z_\alpha\alpha - Z_\delta\delta \end{aligned} \tag{1}$$

in terms of angular rate in pitch $\dot{\theta}$, angle of attack α, and normal acceleration at the center of gravity n. The quantity U_0 is the value of the aircraft's unperturbed velocity and δ represents the elevator deflection. The coefficients M_δ, Z_α, and Z_δ are constants, representative of the flight condition, and have appropriate dimensions.

Angle of attack is difficult to measure, and its interpretation is complicated by turbulence in the air, so, following Shipley,[1] we algebraically eliminate it from the equations and find

$$\begin{aligned} \ddot{\theta} &= \bar{\beta}_q\dot{\theta} + \bar{\beta}_n n + \bar{\beta}_\delta\delta \\ \dot{n} &= -Z_\alpha\dot{\theta} + (Z_\alpha/U_0)n - Z_\delta\dot{\delta} \end{aligned} \tag{2}$$

The first equation in ≡ 2 is the fundamental relation in this study. The C^* quantity is

$$C^* = n + l\ddot{\theta} + U_c\dot{\theta} \tag{3}$$

in which l is the distance of the pilot forward of the center of gravity and U_c is a number called the cross-over velocity, usually taken around 400 fps. If we require C^* to be precisely a multiple of the command input C_c,

$$C^* = -kC_c \tag{4}$$

Equation (3) may be written as

$$\ddot{\theta} = -(U_c/l)\dot{\theta} - (1/l)n - (k/l)C_c \tag{5}$$

This is exactly the form of the fundamental equation in Eq. (2).

The control configuration is diagrammed in Fig. 1. The equations are

$$\begin{aligned} \ddot{\theta} &= \bar{\beta}_q\dot{\theta} + \bar{\beta}_n n + \bar{\beta}_\delta[(C + f)/(T_aS + 1)] \\ \epsilon &= \ddot{\theta} - \beta_q\dot{\theta} - \beta_n n - \beta_\delta[C/(T_aS + 1)] \\ C &= C_c + H_{\ddot{q}}\ddot{\theta} + H_q\dot{\theta} + H_n n \end{aligned} \tag{6}$$

Since the LaPlace operator is equivalent to differentiation, the first two equations may be rewritten as

$$\begin{aligned} (T_aS + 1)\epsilon &= (T_aS + 1)\ddot{\theta} - T_a\beta_q\ddot{\theta} - T_a\beta_n\dot{n} - \beta_n n - \beta_\delta C \\ (T_aS + 1)\ddot{\theta} &= T_a\bar{\beta}_q\ddot{\theta} + \bar{\beta}_q\dot{\theta} + T_a\bar{\beta}_n\dot{n} + \bar{\beta}_n n + \bar{\beta}_\delta(C + f) \end{aligned} \tag{7}$$